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Factorization of the Cover Polynomial*

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Chung and Graham's cover polynomial is a generalization of the factorial rook polynomial in which the second variable keeps track of cycles. We factor the cover polynomial completely for Ferrers boards with either increasing or decreasing column heights. For column-permuted Ferrers boards, we find a sufficient condition

conclude with some conjectures.

1. INTRODUCTION

Chung and Graham's cover polynomial [2] generalizes Goldman, Joichi, and White's "factorial" rook polynomial [6] to two variables. The factorial rook polynomial is $\sum_i r_i x^{\underline{n-i}}$, where r_i is the number of ways to place i non-attacking rooks on some fixed subset of squares of an n by n chessboard. This fixed subset is called the "board." In the cover polynomial, the second variable keeps track of the number of cycles of each rook placement, considered as a partial "covering" of the vertices in the directed graph associated with the board. Goldman, Joichi, and White proved the following factorization of the factorial rook polynomial of a Ferrers board in terms of its column height c_i:

$$\sum_i r_i x^{\underline{n-i}} = \prod_i (x + c_i - i + 1).$$

We investigate the factorization of the cover polynomial for Ferrers boards, and, more generally, for column-permuted Ferrers boards, or "skyline" boards.

For Ferrers boards with increasing column heights the cover polynomial is a product of linear factors; this result, discovered independently by

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Haglund [3], is a straightforward generalization of Goldman Joichi, and White's theorem. When the column heights are decreasing, the cover polynomial still factors completely, but the proof is more complicated. The analogue of Goldman, Joichi, and White's method succeeds when the board has a certain property, which we call "uprightness." We show that we can transform any decreasing Ferrers board into an upright board by "flipping the diagonal hooks," an idea which essentially appears in a paper of Foata and Schüzenberger [4]. We show that the cover polynomial is invariant under this transformation.

For general skyline boards, the cover polynomial does not factor completely; however, we give a sufficient condition for its partial factorization. The idea is to find a set of columns within which the distribution of cycles of rook placements is independent of the rook placements in the other columns. One of the two factors of the cover polynomial is the cover polynomial of the board obtained by deleting this distinguished set of columns; the other factor arises from a kind of restriction to this set of columns, but it is not necessarily the cover polynomial of any skyline board.

We give a sufficient condition for this factor to be a cover polynomial as the first of several applications of the partial factorization theorem. We give a new proof of a factorization theorem of Chung and Graham in the case of column-permuted Ferrers boards. We find that full columns and empty columns within any skyline board correspond to linear factors of the cover polynomial. We present another proof of this, using Chung and Graham's theorem and a reciprocity theorem discovered by Gessel and independently by Chow. We also give alternate proofs that the cover polynomials of increasing and decreasing Ferrers boards factor completely.

The most extensive application involves column permutations of the "staircase" board, the board whose column heights from left to right are $1, 2, ..., n$. For a given column permutation, we define an equivalence relation on the columns whose equivalence classes correspond to factors of the cover polynomial. We also indicate how to draw the functional digraph of the permutation so that the non-intersecting pieces correspond to the equivalence classes. We give a sufficient condition for the complete factorization of such boards. We conclude with some conjectures about permutations that induce a single equivalence class, and also a conjecture about the general applicability of the partial factorization theorem.

2. DEFINITIONS

For integers s and t let $[s, t] = \{i: i \text{ is an integer and } s \leqslant i \leqslant t\}$. Let $[t]$ be an abbreviation for $[1, t]$. We fix an integer n. A *board* is a subset of $[n] \times [n]$. There are two useful geometric interpretations for a board B:

In the first interpretation, $[n] \times [n]$ represents an n by n array of squares. We let the element (i, j) of B identify the square in the ith column of the array, numbered from the left to right, and the jth row of the array, numbered from bottom to top. Thus B is a collection of squares in the array. Each element (i, j) in B is called a *cell*. The *diagonal* of the array is the subset $\{(1, 1), (2, 2), ..., (n, n)\}$.

In the second interpretation, B is a directed graph, or digraph for short, on n vertices. Here, the element (i, j) of B refers to the directed edge from vertex i to vertex j. Note that such a digraph cannot have multiple edges, but it may have loops.

For historical reasons, the default interpretation of a board will be as the collection of squares. To distinguish it explicitly, we may sometimes refer to the digraph as the "associated" digraph of the board. But it will be convenient to switch from one interpretation to the other freely, when the meaning is clear from the context. Thus, for example, a *cycle* of a board is a subset of the board whose elements form a directed cycle in the associated digraph; similarly, a *path* of a board is a subset whose elements form a directed path in the digraph.

A *compatible subset* of a board B is a subset of squares of B such that no two squares are in the same row or column. More formally, if (i, j) and (k, l) are any two distinct elements of a compatible subset, then $i \neq k$ and $j \neq l$. If we place chess rooks in the cells of the subset, then this condition is equivalent to the requirement that no two rooks attach each other; that's why compatible subsets are often called "rook placements" in the literature. For any nonnegative integer i, define the ith *rook number* of B to be the number of compatible i-subsets of B. We denote this number by $r_i(B)$, or just r_i when the board B is clear from the context. So, for example, r_1 is the number of squares in the board. By convention $r_0 = 1$. It is easy to see that $r_i = 0$ for $i > n$.

For example, let B be as in Fig. 1. We get $r_0 = 1$, $r_1 = 5$, $r_2 = 6$, $r_3 = 1$, with the latter two illustrated in Fig. 2.

The *ordinary rook polynomial* of a board B, denoted $R_B(x)$, is defined by $R_B(x) = \sum_i r_i(B)\, x^i$. Again, if the board is understood, the "B" may be dropped from the notation. Note also that we may allow i to range over all integers, because r_i is non-zero for only finitely many values of i.

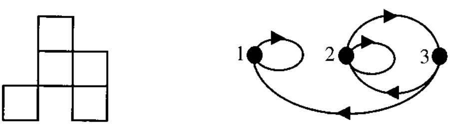


FIG. 1. $n = 3$, $B = \{(1, 1), (2, 2), (2, 3), (3, 1), (3, 2)\}$.

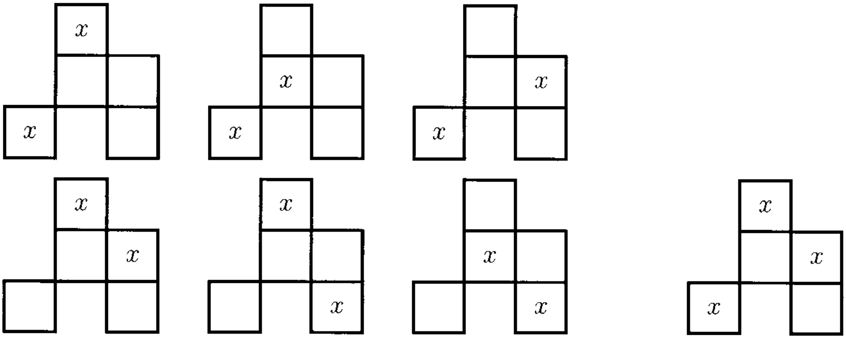


FIGURE 2

Given an integer i, define $x^{\underline{i}}$, the ith falling power of x, and $x^{\overline{i}}$, the ith rising power of x, by $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$ and $x^{\overline{i}} = x(x+1)\cdots(x+i-1)$. Goldman, Joichi, and White defined the *factorial rook polynomial*, denoted $p(B; x)$, by replacing ordinary powers in the ordinary rook polynomial with falling powers and reversing the order of the coefficients. In other words, $p(B; x) = \sum_i r_i(B)\, x^{\underline{n-i}}$.

An *increasing Ferrers board* is a board such that all cells below or to the right of a cell in the board are also in the board; thus, it is composed of adjacent columns whose heights increase (weakly) from left to right. The column heights form a (weakly) increasing sequence of n nonnegative integers, say $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$, and such a sequence determines the board completely. (*Strictly increasing*, *decreasing*, and *strictly decreasing* Ferrers boards are defined in the obvious way.)

Goldman, Joichi, and White [6] showed that the factorial rook polynomial of a Ferrers board factors completely. For the board in Fig. 3,

$$p(B; x) = x^{\underline{4}} + 8x^{\underline{3}} + 14x^{\underline{2}} + 4x^{\underline{1}} + 0 = x^2(x+1)^2.$$

One way to generalize the factorial rook polynomial is to consider the number of cycles of each rook placement. For $i, j \in [n]$, let $r_{i,j}(B)$, or $r_{i,j}$ for short, be the number of compatible i-subsets of B which contain exactly

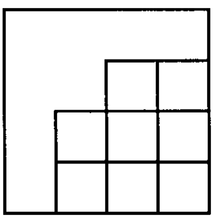
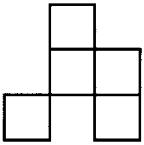
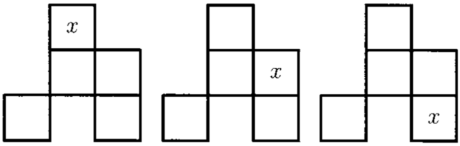
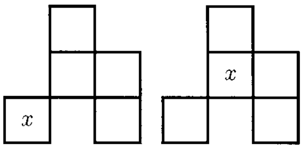
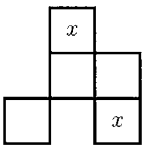


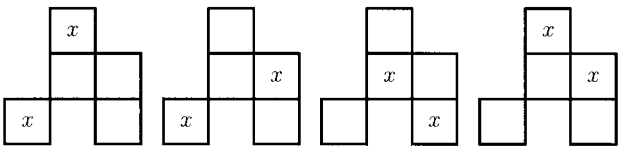
FIG. 3. $n = 4$, $c_1 = 0$, $c_2 = 2$, $c_3 = 3$, and $c_4 = 3$.

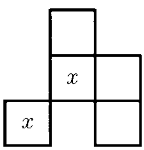
$r_{0,0} = 1$ 

$r_{1,0} = 3$ 

$r_{1,1} = 2$ 

$r_{2,0} = 1$ 

$r_{2,1} = 4$ 

$r_{2,2} = 1$ 

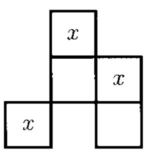
$r_{3,2} = 1$ 

FIGURE 4

j cycles. These are the *cycle rook numbers*. Note that the number of cycles is well-defined: any compatible set of edges in the digraph can be uniquely partitioned into maximal directed paths and directed cycles; because each vertex will have indegree and outdegree not exceeding one. We define the *cover polynomial* of the corresponding digraph D, denoted $C(D; x, y)$, or $C(D)$ for short, by $C(D; x, y) = \sum_{i,j} r_{i,j}(B)\, x^{\underline{n-i}} y^j$. Note that $C(D; x, 1) = p(B; x)$. Chung and Graham [2] originally define the cover polynomial recursively on two sorts of digraphs; $D \backslash e$, formed by deleting the directed edge e, and D/e, formed by contracting e and its endpoints to a single vertex. Specifically, when D has i vertices and no edges, they define $C(D; x, y)$ to be $x^{\underline{i}}$; otherwise, they apply the following recursion:

$$C(D) = \begin{cases} C(D \backslash e) + yC(D/e) & \text{if} \quad e \text{ is a loop} \\ C(D \backslash e) + C(D/e) & \text{if} \quad e \text{ is not a loop.} \end{cases}$$

They prove that

$$C(D; x, y) = \sum_{i,j} C_D(i, j)\, x^{\underline{i}} y^j,$$

where $C_D(i, j)$ is the number of ways of disjointly "covering" all the vertices of D with i maximal directed paths and j directed cycles. The number i may include some trivial paths, each of which covers a single vertex. They also prove that $\sum_j C_D(n-i, j) = r_i(B)$. We sketch their proof here in order to show a little more: $C_D(n-i, j) = r_{i,j}$, which implies that our definition of $C(D; x, y)$ is equivalent to theirs.

Any compatible i-subset of B corresponds to a partial covering J of the vertices of D with i directed edges, which form, say, r maximal paths and j cycles. Each cycle contains as many vertices as edges, but each directed path contains an extra vertex; therefore J covers $i + r$ vertices. Adding $n - i - r$ trivial paths to cover the remaining vertices yields a full covering in which the total number of paths is $n - i$. This proves the claimed result.

In Fig. 4 we illustrate the cycle rook numbers of the board B from Figs. 1 and 2. Thus, $C(B; x, y) = x^{\underline{3}} + (3 + 2y)\, x^{\underline{2}} + (1 + 4y + y^2)\, x^{\underline{1}} + y^2$.

3. FACTORIZATION OF THE COVER POLYNOMIAL OF FERRERS BOARDS

In order to factor the cover polynomial of a Ferrers board, let us review Goldman, Joichi, and White's combinatorial proof that the factorial rook polynomial of a Ferrers board factors completely.

Let B be a Ferrers board. Let x be a positive integer, and construct a new board $B^{\#}$ by adjoining x full new rows to B. We can draw these rows beneath B, as illustrated in Fig. 5, and consider them to be indexed by the elements of $[n+1, n+x]$.

The factorial rook polynomial counts the number of ways of placing n non-attacking rooks on $B^{\#}$. To see this, we place i rooks on the original board B in $r_i(B)$ ways. This leaves $n-i$ empty columns, each of which must get a rook in one of the extra x rows. If we place these rooks one column at a time, say from left to right, then there will be x rows available for the first empty column. After we choose one of those rows, there will be $x-1$ rows available for the rook in the next empty column. Continuing in this manner, we find that there are $r_i(B)\, x^{\underline{n-i}}$ placements of n rooks on $B^{\#}$ with exactly i rooks in B; considering all possible i, we get $\sum_i r_i(B)\, x^{\underline{n-i}}$ placements of n rooks on $B^{\#}$, as claimed.

We can also count the number of ways of placing n non-attacking rooks on $B^{\#}$ without distinguishing between the original board and the extra columns. We simply choose a row for each of the n columns, again, one at a time, from left to right. Because B is a Ferrers board, any placement of a rook must eliminate one row from each succeeding column. This means that there are $x+c_1$ choices for column 1, $x+c_2-1$ choices for column 2, and, in general, $x+c_i-i+1$ choices for column i. We equate this product with the previous count. Because the resulting polynomial identity is true for infinitely many values of x, it must be true for all values of x. This proves that the factorial rook polynomial factors completely:

THEOREM [Goldman, Joichi, and White] 3.1. *Let $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_k$ be the sequence of column heights of a Ferrers board B. Then*

$$p(B; x) = \prod_{i=1}^{n} (x + c_i - i + 1).$$

Note that the same proof works for decreasing Ferrers boards, as long as we place rooks in columns from right to left instead of left to right. Indeed, the ordinary rook numbers are invariant under any permutation of rows or columns, so we can permute the columns of a Ferrers board arbitrarily without changing the factorial rook polynomial.

However, the cycle rook numbers are not invariant under column permutations. For example, the cycle rook number $r_{1,1}$ is simply the number of columns that hit the diagonal, and it clearly can vary, as illustrated in Fig. 6.

This means that we must consider each column permutation separately. Although the cover polynomial does not factor completely for arbitrary column permutations of Ferrers board, it does factor completely for both increasing and decreasing Ferrers board. We give an example in Fig. 7.

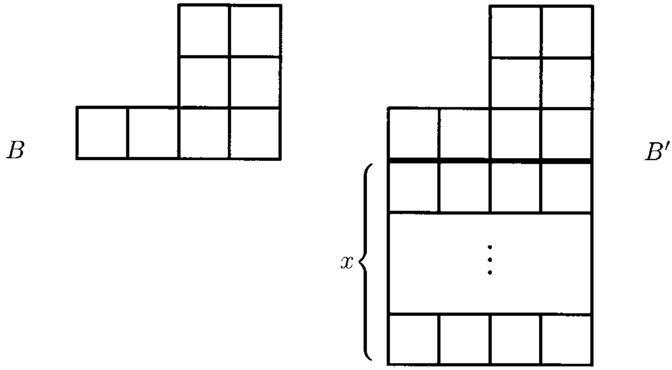


FIG. 5. An example of an extended board with $n = 4$.

For increasing Ferrers boards, Haglund independently discovered this result, and a q-version appears in a paper of Ehrenborg, Haglund, and Readdy [3]. The proof is a straightforward generalization of the proof of Theorem 3.1:

THEOREM 3.2. *Let $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$ be the sequence of column heights of a Ferrers board B. Then*

$$C(B; x, y) = \prod_{c_i \geqslant i} (x + c_i - i + y) \prod_{c_i < i} (x + c_i - i + 1).$$

Proof. As in Theorem 3.1, it will suffice to prove the result for an arbitrary positive integer x. Let $B^{\#}$ be the board formed by adjoining x extra rows below B. We again employ two methods for counting the number of ways of placing n rooks on $B^{\#}$, giving a weight of y^j to each placement which contains exactly j cycles in B. The first method is to analyze placements with exactly i rooks on B. Then the sum of weights of such placements will be $\sum_j r_{i,j}(B)\, y^j$, and there will be $x^{\underline{n-i}}$ ways to place the remaining rooks. Summing over all i gives the cover polynomial.

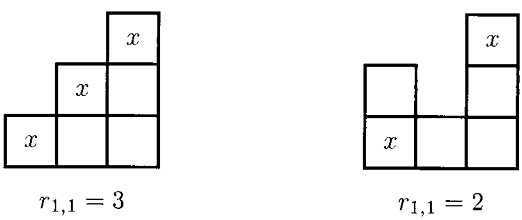


FIGURE 6

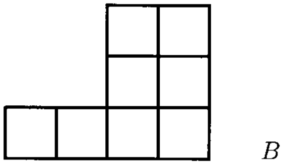


FIG. 7. $n=4$, $C(B; x, y) = (x+y)\, x(x+y)\, x$.

The second method is to place a rook in each column of $B^{\#}$, proceeding from left to right. We weight each rook that completes a cycle in B by y, we weight the other rooks by 1, and we weight the entire placement by the product of the weights of the n rooks. In this way, a placement with j cycles gets the weight y^j, as required. Moreover, since B is a Ferrers board, this method ensures that there will be $x+c_i-i+1$ rows available in the ith column, independent of previous choices. We show that in fact the sum of the weights of the cells in the column is independent of previous choices; this means that the sum of the weights of all placements is the product of the sums of the weights of the cells in each individual column.

If $c_i \geqslant i$, then exactly one choice for the placement of the rook in the ith column completes a cycle, for any given set of choices in the previous $i-1$ columns. To see this, consider the associated digraph: if no previous directed edge goes into vertex i, then choosing (i, i) completes a cycle. If there exists a directed edge into i, then there must be a directed path into i originating at some vertex v. Since $v < i \leqslant c_i$, we may choose the cell (i, v) to complete the cycle. It is clear that no other edge emanating from vertex i completes a cycle. We conclude that the sum of the weights of the cells in the column is $x+c_i-i+y$.

On the other hand, if $c_i < i$, then $c_k < i$ for all $k \leqslant i$, which means that the indegree of vertex i is zero at this stage; therefore, the choice of the edge emanating from vertex i cannot complete a cycle. In this case, the sum of the weights of the cells in the column is $x+c_i-i+1$. ∎

For a decreasing Ferrers board, one would naturally want to imitate the previous proof. We would need to choose a cell from each column, proceeding from right to left, to ensure that each rook eliminated a row in each succeeding column. We would want to claim that, if $c_i \geqslant i$, the there is exactly one placement of a rook in column i which completes a cycle at that stage. However, this is not quite true. Consider the board in Fig. 8.

Suppose we choose the cell $(3, 2)$ in the third column. Even though $c_2 \geqslant 2$, the only cell in the second column which could complete a cycle, $(2, 3)$, is not in the board. To ensure that columns which reach the diagonal avoid this problem, we require the board to have a certain "upright" quality. In particular, define a board B to be *upright* if and only if whenever (j, i) is a cell of B with $i < j$, then (i, j) is also a cell of B.

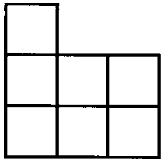


FIG. 8. $n=3$.

THEOREM 3.3. *Let $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_n$ be the sequence of column heights of an upright decreasing Ferrers board B. Then*

$$C(B; x, y) = \prod_{c_i \geqslant i} (x + c_i - (n-i) + y - 1) \prod_{c_i < i} (x + c_i - (n-i)).$$

Proof. We use the same method as Theorem 3.2, except here we choose cells from right to left; it will suffice to analyze the sums of the weights of the columns. We claim that, as before, the ith column contains exactly one cell which completes a cycle, given precious choices, if $c_i \geqslant i$, and no such cell otherwise.

Consider the corresponding digraph. At this stage, we have chosen edges emanating from the vertices $n, n-1, ..., i+1$. This means that if $c_i < i$, then each possible edge from vertex i terminates at a vertex which has outdegree 0, so that choice could not complete a cycle.

Conversely, suppose $c_i \geqslant i$. If no previous directed edge terminates at vertex i, then choosing (i, i) completes a cycle. If there exists a directed edge terminating at vertex i, then there must be a directed path into i originating at some vertex v. Let $v = v_1, v_2, ..., v_m = i$ be the sequence of vertices in the path; then (v_k, v_{k+1}) is a previously chosen directed edge for $k \in [m-1]$, and of course $v_k \geqslant i$ for $k \in [m]$. We take m to be maximal, so that indegree $(v) = 0$; it remains to show that we can choose the cell (i, v), completing a cycle.

Suppose $k \in [m]$; we prove by induction on k that $c_i \geqslant v_k$. In this induction we descend from $k = m$ to $k = 1$. We already have the initial case $c_i \geqslant i = v_m$, so suppose $1 \leqslant k < m$. By induction, $v_{k+1} \leqslant c_i$, so if $v_k < v_{k+1}$, then clearly $v_k \leqslant c_i$. On the other hand, if $v_k > v_{k+1}$, then because B is upright, (v_{k+1}, v_k) is a cell in B, so that $c_i \geqslant c_{v_{k+1}} \geqslant v_k$, completing the induction. In particular, $c_i \geqslant v_1 = v$. ∎

To show that the cover polynomial of an arbitrary decreasing Ferrers board factors completely, we give a straightforward procedure for transforming it into an upright decreasing Ferrers board which has the same cover polynomial. The basic idea goes back to a paper of Foata and Schützen berg [4]. Figure 9 gives an example with $n=7$.

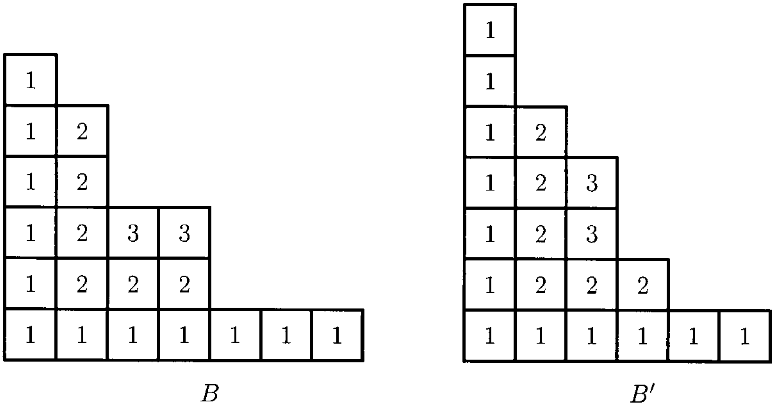


FIG. 9. $C(B; x, y) = C(B'; x, y)$ and B' is upright.

The ith *diagonal hook* of a board B is the set of all $(j, k) \in B$ such that either $j = i$ and $k \geqslant i$, or $k = i$ and $j \geqslant i$. In Fig. 9, the numbers in the cells identify the diagonal hooks; B can be transformed into B' by "flipping" the first and third diagonal hooks across the diagonal. In general, the procedure is to flip the diagonal hooks which, considered as sub-boards, are not upright.

Given a decreasing Ferrers board B containing m cells of the diagonal, here is the formal construction of B'. Let H_i be the ith diagonal hook. Let u_i and v_i be the height and width of H_i, respectively. Note that $u_1 > u_2 > \cdots > u_m$ and $v_1 > v_2 > \cdots > v_m$. For each $i \in [m]$, we define $\tilde{H}_i$, the *flip* of H_i, to be the set of all (j, k) such that $(k, j) \in H_i$. Clearly $\tilde{H}_i$ is itself a hook, and, considered as sub-boards, $\tilde{H}_i$ is upright if H_i is not upright. For each $i \in [m]$, we define H'_i as

$$H'_i = \begin{cases} H_i & \text{if} \quad H_i \text{ if upright} \\ \tilde{H}_i & \text{if} \quad H_i \text{ is not upright,} \end{cases}$$

and we let u'_i and v'_i be the height and width of H'_i. We define B' to be the union of $H'_1, H'_2, ..., H'_m$.

LEMMA 3.4. *If B is a decreasing Ferrers board, then B' is an upright decreasing Ferrers board.*

Proof. We use the notation defined above. By construction, H'_i is upright for each i, therefore B' is upright. We show that B' is a decreasing Ferrers board by induction on m.

When $m = 0$, both B and B' are empty, and the result is clear. Suppose $m > 0$. Let $A = B \backslash H_1$, and let $A' = B' \backslash H'_1$. As a subset of $[2, n] \times [2, n]$, A

is a decreasing Ferrers board with $m-1$ hooks, so by induction A' is also a decreasing Ferrers board. It remains only to show $v_1' > v_2'$, so that B' is a Ferrers board, and $u_1' > u_2'$, so that B' is decreasing. If both H_1 and H_2 are upright, or if neither is upright, then the two inequalities are immediate. Suppose H_2 is upright and H_1 is not upright; in other words, $u_2 \geqslant v_2$ and $v_1 > u_1$. Then $u_1' = v_1 > u_1 > u_2 = u_2'$ and $v_1' = u_1 > u_2 \geqslant v_2 = v_2'$, as desired. The other case, in which H_2 flips but H_1 does not, is similar. ∎

It will follow quickly from the next lemma that the cover polynomial is invariant under this transformation.

LEMMA 3.5. *Let B be a decreasing Ferrers board, and let $\mathscr{P}$ and $\mathscr{P}'$ be the sets of simple directed paths and cycles in the digraphs corresponding to B and B'. There exists a bijection*

$$\phi \colon \mathscr{P} \to \mathscr{P}'$$

such that, for any $P \in \mathscr{P}$, the following properties hold:

(1) *the set of vertices in $\phi(P)$ is the same as the set of vertices in P*;

(2) *$\phi(P)$ is a cycle if and only if P is a cycle.*

Proof. We continue to use the notation defined in the construction of B' and in the proof of the Lemma 3.4. Again we proceed by induction on m. If $m=0$, then both B and B' are empty; hence, both $\mathscr{P}$ and $\mathscr{P}'$ are the set of paths that consist of no edges. Such paths contain either no vertices or one vertex from $[n]$, so it suffices to let ϕ be the identity mapping.

Now suppose $m > 0$. By induction, there exists a bijection ψ from paths and cycles of A to paths and cycles of A' with the two properties stated in the lemma. If vertex 1 is not in P, then $\psi(P)$ is defined, and we let $\phi(P) = \psi(P)$.

If vertex 1 is in P, then there are four cases: P is a path whose initial vertex is 1, P is a path whose terminal vertex is 1, P is a path that contains vertex 1 in its interior, or P is a cycle that contains vertex 1. We regard any path as the sequence of its vertices, ordered from the initial vertex to the terminal vertex. Similarly, we regard a cycle containing vertex 1 as a path from vertex 1 to itself. In this way, we can denote P by concatenating subsequences of its vertices. In particular, the four cases take the forms $1\,P_1$, $P_1\,1$, $P_1\,1\,P_2$, and $1\,P_1\,1$. Note that, other than vertex 1, which appears twice in each cycle, the vertices within any of these sequences are distinct, because the lemma requires paths and cycles to be simple. We can now define $\phi(P)$ according to the orientation of H_1, as indicated in the following table.

	H_1 upright	H_1 not upright
$\phi(1\ P_1) =$	$1\ \psi(P_1)$	$\psi(P_1)\ 1$
$\phi(P_1\ 1) =$	$\psi(P_1)\ 1$	$1\ \psi(P_1)$
$\phi(P_1\ 1\ P_2) =$	$\psi(P_1)\ 1\ \psi(P_2)$	$\psi(P_2)\ 1\ \psi(P_1)$
$\phi(1\ P_1\ 1) =$	$1\ \psi(P_1)\ 1$	$1\ \psi(P_1)\ 1$

We verify that $\phi(P)$ is indeed a path or cycle in B'. Any edge in a path in the image of ψ is automatically in B', so we need only check that the edges containing vertex 1 are in B'. If P_1 or P_2 is either empty or a single vertex, then, by the two properties of the lemma, ψ acts on it as the identity. Thus, for example, if for some vertex z, $P = z1$, and if H_1 is not upright, then according to the table $\phi(P) = 1z$. We know that B' contains the edge $(1, z)$, because B contains the edge $(z, 1)$ and H_1 flips. A similar argument holds for the other entries in the table when P_1 or P_2 is a singleton. If P_1 or P_2 is empty, then the only relevant entries are in the fourth row, and here we know $(1, 1)$ is an edge in B' because $(1, 1)$ is an edge in B. Note also that empty paths cause the domains of the first three rows to overlap somewhat, but in those instances the values of ϕ agree.

We suppose now that P_1 (or P_2) contains two or more vertices. This means that each vertex in P_1 must be incident with an edge in A. We again illustrate the argument in the case where H_1 is not upright and $P = P_1\ 1$. Let w be the initial vertex in $\psi(P_1)$; we need to verify that $(1, w)$ is an edge in B'. Suppose it is not. Since H_1 flips, $(w, 1)$ cannot be an edge in B. This means that w is not the initial vertex of any edge in A, since B is a Ferrers board. In particular w is not the initial vertex of any edge in P_1. But ϕ satisfies property 1 of the lemma, and w is a vertex of $\psi(P_1)$, so w must also be a vertex of P_1. Therefore w is the terminal vertex of P_1. But then, because $P = P_1\ 1$, it follows that the edge $(w, 1)$ is in B, contradicting the initial assumption. A similar argument holds for the other entries in the table.

The two properties in the lemma follow directly from the construction of ϕ. It is also straightforward to check that if we replace ψ with ψ^{-1} in the definition of ϕ, then the mapping that results is the inverse of ϕ, so ϕ is a bijection. ∎

THEOREM 3.6. *Given a decreasing Ferrers board B, there exists an upright decreasing Ferrers board B' with the same cover polynomial.*

Proof. Let B' be as defined above, and let ϕ be the bijection given by Lemma 3.5. Any compatible subset J of B can be decomposed into maximal paths and cycles whose vertex sets will be disjoint. When we apply ϕ to these components of J, we get a set of maximal paths and cycles of B'

whose vertex sets are also disjoint, by property 1 of the Lemma 3.5. Hence, these paths and cycles are the components of a compatible subset J' of B'. In this way, ϕ induces a bijection from compatible subsets of B to compatible subsets of B'; moreover, the number of edges and the number of cycles is preserved, by properties 1 and 2 of Lemma 3.5. Therefore, $C(B; x, y) = C(B'; x, y)$. ∎

COROLLARY 3.7. *The cover polynomial of any decreasing Ferrers Board is a product of linear factors.*

To illustrate this, consider again the boards in Fig. 9. The cover polynomial of B is equal to the cover polynomial of B' by Theorem 3.6. The column heights of B' are $c_1 = 7$, $c_2 = 5$, $c_3 = 4$, $c_4 = 2$, $c_5 = 1$, $c_6 = 1$, and $c_7 = 0$, so according to Theorem 3.3, $C(B')$ equals

$$(x+6+y-6)(x+4+y-5)(x+3+y-4)(x+2-3)$$
$$\times (x+1-2)(x+1-1)\, x = x^2(x-1)^2\,(x-1+y)^2\,(x+y).$$

4. PARTIAL FACTORIZATION OF THE COVER POLYNOMIAL OF SKYLINE BOARDS

A *column-permuted Ferrers board*, or *skyline board*, is obtained by rearranging the columns of a Ferrers board. Like an ordinary Ferrers board, a skyline board is uniquely determined by its sequence of nonnegative integer column heights $c_1, c_2, ..., c_n$, except now the column heights are arbitrary. Thus, for a skyline board B, $(i, j) \in B$ if and only if $j \leqslant c_i$. For a permutation σ and a skyline board B, we define the *column permutation of B by σ*, denoted $\sigma(B)$, to be $\{(i, j) : (\sigma(i), j) \in B\}$. This means that if $c_1, c_2, ..., c_n$ are the column heights of B, then for any $i \in [n]$ the ith column of $\sigma(B)$ has height $c_{\sigma(i)}$.

Similarly, we can let the *row permutation of B by σ* be $\{(i, j) : (i, \sigma(j)) \in B\}$, but this is equivalent to a column permutation of B, provided we relabel the vertices appropriately. In particular, when we replace each vertex $v \in [n]$ by $\sigma(v)$, the row permutation of B by σ becomes the column permutation $\sigma^{-1}(B)$:

$$\{(\sigma(i), \sigma(j)) : (i, \sigma(j)) \in B\} = \{(i, j) : (\sigma^{-1}(i), j) \in B\} = \sigma^{-1}(B).$$

Therefore we do not consider row permutation hereafter.

We have seen that the cover polynomial of an increasing or decreasing Ferrers board factors completely. We have also seen that permuting columns can affect the cycle rook numbers, which of course would change the cover polynomial. Indeed, the cover polynomial of a skyline board will not factor completely, in general. Consider the example in Fig. 10 with $n = 3$.

The quadratic factor is irreducible. Let us examine why the method of Goldman, Joichi, and White that we used in the previous theorems does not yield a complete factorization of this board. We once again adjoin x new rows to the board and interpret the cover polynomial as the number of compatible 3-subsets, weighted by cycles, of this extended board. We choose a cell from column 3, then one from column 1, and then one from column 2, with $x+1$ rows available in each case. Although none of the cells in column 3 could possibly complete a cycle, $x+1$ is not a factor of the cover polynomial, and although both column 1 and column 2 reach the diagonal, only column 2 corresponds to a factor of $x+y$. This is because the weights of the choices in column 1 depend on which cell is chosen from column 3; together, the possible pairs of cells from columns 1 and 3 correspond to the quadratic factor in the cover polynomial. In particular, if $(3, 1)$ is chosen, then no cell in column 1 can complete a cycle, but if any of the other x cells in column 3 is chosen, then $(1, 1)$ is available to complete a cycle, and of course none of the other x compatible cells completes a cycle. The sum of the weights of all the possible pairs from these two columns is therefore $1(x+1)+x(x+y)=x^2+xy+x+1$.

By contrast, exactly one of the cells in column 2 will complete a cycle, independent of the choices in the other two columns. This is because column 2 is full, so that in the associated digraph there is an edge from vertex 2 to each of the other vertices; in particular, there is an edge to the initial vertex of the path terminating at vertex 2. This suggests that any column of height n will contribute a factor of the form $x+l+y$ to the cover polynomial, for some $l \in [0, n]$, and similarly that any column of height 0 will contribute a factor of the form $x-l$, and we shall soon confirm this. In general, however, it is not so easy to tell when a choice or set of choices is independent of other choices. Theorem 4.1 will give a sufficient condition for this independence and hence for the partial factorization of the cover polynomial.

We prepare to state the partial factorization theorem by defining a few key notions. We suppose throughout that B is a skyline board whose corresponding digraph is D. In the proofs in the previous section, we

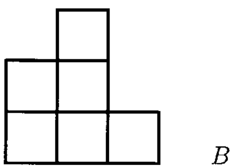


FIG. 10. $C(B)=(x+y)(x^2+xy+x+1)$.

picked cells one at a time from columns in weakly increasing order of height so that we could keep track of the number of rows available. In those cases it was obvious how to pick the columns: left to right for increasing Ferrers boards and right to left for decreasing Ferrers boards. Similarly, for skyline boards we want an ordering of the columns in which the heights are nondecreasing. In particular, we call a permutation τ on n letters a *consistent ordering* (of the columns) of B if $i<j$ implies that $c_{\tau(i)} \leqslant c_{\tau(j)}$. Thus, for example, column $\tau(1)$ is a column of minimum height. Every skyline board has a consistent ordering, and it will be unique if and only if the column heights are all distinct.

The factorization will be described recursively in terms of two sorts of subdigraphs. Let $D\backslash V$ be the digraph formed by deleting from D all of the vertices of V and all of the edges incident with vertices of V. Let $D \mid V$ be the subdigraph induced by the directed edges whose initial vertices are in V. Note that $D \mid V$ may include vertices that are not in V, as terminal vertices, but it does not include any isolated vertices that are not in V.

Now we describe the sufficient condition for the factorization. Let τ be a consistent ordering of B. Define a subset V of the vertices of D to be *consecutive with respect to* τ if there exist $a, k \in [n]$ such that $V = \tau([a, a+k-1])$. We define the *predecessors of* V, denoted $V_<$, to be $\tau([a-1])$ and the *successors of* V, denoted $V_>$, to be $\tau([a+k, n])$. Consequently, when we use Goldman, Joichi, and White's method, the edges emanating from the vertices of $V_<$ will be chosen before those of V, which in turn will be chosen before those $V_>$. Define a subset V of the vertices of D to be *independent with respect to* τ if the following three conditions hold: V is consecutive with respect to τ, no edge emanating from a vertex of $V_<$ terminates at a vertex of V, and there is an edge from every vertex of $V_>$ to every vertex of V. In other words, for all $u \in V_<$, $v \in V$, and $w \in V_>$, we have $(u, v) \notin B$ and $(w, v) \in B$.

THEOREM 4.1. *Let D be the digraph of a skyline board B, let τ be a consistent ordering of D, and let V be a subset of the vertices of D that is independent with respect to τ. Then $C(D) = C(D\backslash V)C^*(V)$, where $C^*(V)$ is a polynomial which can be computed from the subdigraph $D \mid V$.*

It will suffice to prove the formula for an arbitrary positive integer x, as in Theorems 3.1–3.3. Let $D^\#$ be the digraph corresponding to $B^\#$; in other words, we adjoin the x elements in $[n+1, n+x]$ to the vertices of D, and we adjoin edges from each vertex in $[n]$ to each extra vertex. Of course, these "destination vertices" correspond to x extra rows below B. Moreover, each compatible n-subset of edges corresponds to a covering of $[n]$ in $D^\#$. As before we count these in two different ways. First, we weight each such covering by y^j, where j is the number of cycles in the covering; the analysis used in Theorem 3.2 shows that the sum of all the weights is the cover polynomial. We get a second count by choosing edges originating from those n vertices, one at a time, according to some consistent ordering, and weighting the last edge chosen in each cycle by y.

The calculation of the cover polynomial in the discussion of Fig. 10 illustrates the latter type of count. Moreover, the quadratic factor there is an example of $C^*(V)$; we calculated it by keeping track of the various choices and their consequences for the cycle structure. In order to get a formal description of the process, we will encode the choices as nodes in a certain tree. Note that a "node" is always in the tree, while a "vertex" is always in a digraph. Similarly, to make it easier to distinguish the two settings, we refer to paths in the tree as "tree-paths," and we label nodes with Greek letters instead of the Roman letters that we use for unknown vertices in the digraph.

The *choice tree associated with the digraph* is a rooted, labeled tree that depends on $D^{\#}$, including a particular value of x, and also on a consistent ordering τ; we denote it simply by T. Here is the construction. We let ρ be the root of T, and we let T have $n+1$ levels, from level 0, the root, to level n, the leaves. We let the number of children of ρ equal the outdegree of $\tau(1)$, and we label them with the distinct edges emanating from $\tau(1)$. We construct and label the remaining nodes one level at a time, as follows: given a node in level i of the tree, labeled by an edge emanating from $\tau(i)$, we let the number of its children be i less than the outdegree of $\tau(i+1)$. We label each child with a distinct edge emanating from $\tau(i+1)$ that is compatible with the edges that label its ancestors in the tree. This accounts for all such edges, because each of those ancestors (except ρ) eliminates one of the possible terminal vertices for $\tau(i+1)$, by the consistency of τ. Of course, the process ends at level n, when we choose the edges emanating from $\tau(n)$. In this way, the coverings of $[n]$ in $D^{\#}$ will be in one-to-one correspondence with the set of tree-paths from ρ to a leaf.

Let us reconsider the example in Fig. 10 in light of these definitions. Suppose that $x=1$ and that the extra vertex is vertex 4 in the digraph $D^{\#}$ corresponding to the board in Fig. 10. Recall that (i, j) is the edge from i to j. In Fig. 11 we depict the digraph and its choice tree.

In Fig. 12, we illustrate the bijection from the set of $[n]$ in $D^{\#}$ to the set of tree-paths from the root to a leaf with an example of the correspondence. The node labeled $(2,1)$ has weight y because the edge from 2 to 1 completes a cycle in the covering. In general, a node has weight y if its corresponding edge in the digraph completes a cycle with the edges corresponding to a subset of the node's ancestors in the tree; otherwise the weight of the node is 1.

We can see in Fig. 11 that the children of the root correspond to the possible destinations for 3, the grandchildren to the destinations for 1, and

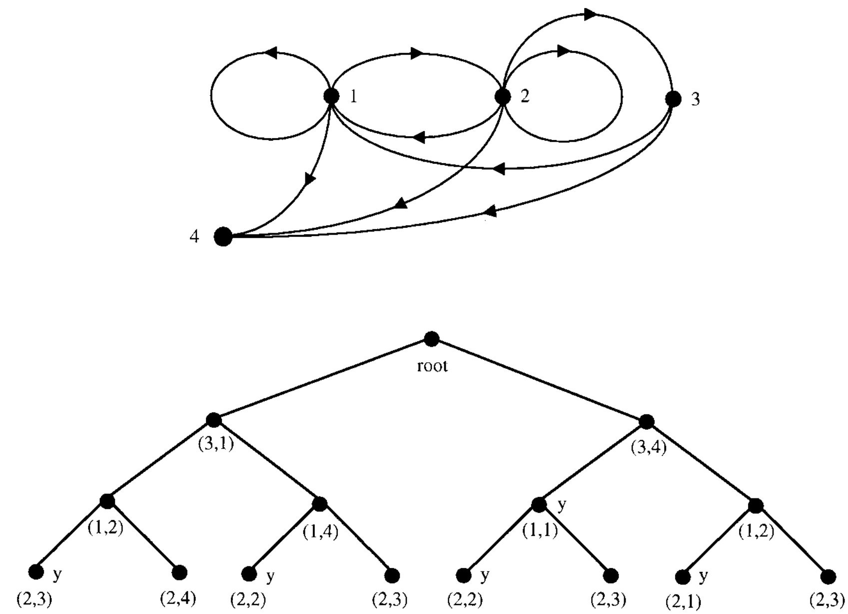


FIG. 11. The digraph $D^{\#}$ and its choice tree T.

the great-grandchildren to the destinations for 2; this is a consistent ordering of the vertices, because $c_3 \leqslant c_1 \leqslant c_2$ in the skyline board in Fig. 10. Moreover, $V = \{1, 3\}$ is independent with respect to this ordering, with $V_<$ empty and $V_> = \{2\}$, so Theorem 4.1 applies. The boards of $D \backslash V$ and $D \mid V$ and the factors they determine are shown in Fig. 13.

For $D \backslash V$, note that $n = 1$, because we deleted two of the three vertices, so $C(D \backslash V)$ is linear in x. It then follows from Theorem 4.1 that $C^*(V)$ is quadratic in x. It would be nice if $C^*(V)$ were the cover polynomial of $D \mid V$; unfortunately, its board has three columns, so its cover polynomial is cubic in x, not quadratic. In fact, this particular $C^*(V)$ is not the cover polynomial of any board. For suppose $C(B_0) = x^2 + xy + x + 1$ for some board B_0. Since $C^*(V)$ is quadratic in x, each of the linear terms

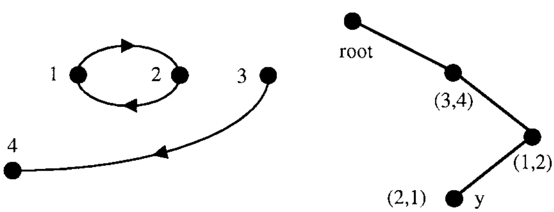


FIG. 12. A covering and its corresponding tree-path.

$D \setminus V = \{(2,2)\}$ $\qquad$ $D \mid V$

$C(D \setminus V) = x + y$ $\qquad$ $C^*(V) = x^2 + xy + x + 1$

FIGURE 13

corresponds to the placement of a single rook. In particular, xy represents a rook on the diagonal, x represents a rook off the diagonal, and there are no other rooks. If the two rooks are compatible, then the constant term is y; if they are not compatible, then the constant term is 0. In either case, the constant term cannot be 1. In Corollary 6.1, we will give a sufficient condition for $C^*(V)$ to be the cover polynomial of a modification of $D \mid V$.

5. PROOF OF THE PARTIAL FACTORIZATION THEOREM

To facilitate the proof of the theorem, we assign weights to nodes and three-paths and subtrees in T. As in the above example, a node has weight y when the edge that labels it completes a cycle with edges labeling its ancestors in the tree; otherwise, the node has weight 1. We weight each tree-path by the product of the weights of its nodes. Because each cycle is counted exactly once, each tree-path from ρ to a leaf has the same weight as the corresponding covering of $[n]$ in $D^{\#}$. We weight the entire tree T by the sum of the weights of all of the tree-paths from the children of ρ to a leaf. We denote these various weight functions by wt. Naturally, T is constructed so that $\mathrm{wt}(T) = C(D)$. Furthermore, for an arbitrary rooted subtree T' of T we can let $\mathrm{wt}(T')$ be the sum of the weights of the tree-paths from the children of the root of T' to the leaves of T'.

We repeat for emphasis an unusual feature of these definitions: the root of a subtree does not contribute to the weight of the subtree; however, the initial node of a tree-path does contribute to the weight of a tree-path.

Let L_i be the set of nodes in level i of T. Let $p(\mu)$ denote the parent of node μ. Let $P(v, v')$ denote the tree-path from node v to node v'. Given $m \in \mathbb{N}$ and a node v, let $T(v, m)$ be the subtree of T rooted at v, with m full levels below v. Recall the meaning of the two integers a and k associated with the given set of independent vertices V: $a = \min\{i : \tau(i) \in V\}$, and $k = |V|$. Thus, if $\mu \in L_{a-1}$, then $T(\mu, k)$ encodes the possible ways to choose the edges emanating from the vertices of V.

We can now formally define the value of $C^*(V)$ at x to be $\mathrm{wt}(T(\mu, k))$, where $\mu \in L_{a-1}$. We can see from the definitions of the weight functions that this yields a polynomial function in x and y. Lemma 5.1 below shows that $C^*(V)$ is independent of the choice of μ. In fact, because Theorem 4.1 holds for any consistent ordering τ, and because $C(D)$ and $C(D\backslash V)$ are independent of τ, $C^*(V)$ is also independent of τ. This means that the value of $C^*(V)$ at $x+a-1$ is the weight of a choice tree of $D|V$ with x extra vertices. The notation $C^*(V)$ is a bit misleading because it does not reflect the dependence on D, but we commit this abuse for the sake of brevity; we presume that D will be clear from the context.

We will use three lemmas in the proof of Theorem 4.1. Lemma 5.1 says that the cycle structure of the edges emanating from the vertices of V is independent of the choices for the destinations of the predecessors of V. Lemma 5.2 says that for a given set of choices for the destinations of the predecessors of V, the cycle structure of the edges emanating from the successors of V is independent of the choices for the destinations of the vertices of V. Lemma 5.3 amplifies this: the cycle structure of the edges emanating from the predecessors and successors together is independent of the choices for the destinations of V.

Lemma 5.1-5.3 are all subject to the conditions of Theorem 4.1, and they use the notation introduced in this section.

LEMMA 5.1. *If μ and μ' are in L_{a-1}, then* $\mathrm{wt}(T(\mu, k)) = \mathrm{wt}(T(\mu', k))$.

LEMMA 5.2. *If μ is in L_{a-1}, and ω and ω' are leaves of $T(\mu, k)$, then*

$$\mathrm{wt}(T(\omega, n+1-a-k)) = \mathrm{wt}(T(\omega', n+1-a-k)).$$

LEMMA 5.3. *For each μ in L_{a-1}, let ω_μ be any leaf of $T(\mu, k)$. Then*

$$\sum_{\mu \in L_{a-1}} \mathrm{wt}(P(\rho, \mu))\, \mathrm{wt}(T(\omega_\mu, n+1-a-k)) = C(D\backslash V).$$

The proofs of these lemmas are somewhat technical, so we first prove the theorem itself.

Proof of Theorem 4.1. By the definitions of the weight functions, we have that

$$\begin{aligned} C(D) &= \sum_{\text{leaves } \lambda \text{ of } T} \mathrm{wt}(P(\rho, \lambda)) \\ &= \sum_{\mu \in L_{a-1}} \mathrm{wt}(P(\rho, \mu))\, \mathrm{wt}(T(\mu, n+1-a)) \\ &= \sum_{\mu \in L_{a-1}} \mathrm{wt}(P(\rho, \mu)) \sum_{\text{leaves } \omega \text{ of } T(\mu, k)} \frac{\mathrm{wt}(P(\mu, \omega))}{\mathrm{wt}(\mu)} \\ &\quad \times \mathrm{wt}(T(\omega, n+1-a-k)). \end{aligned}$$

Now for each $\mu \in L_{a-1}$ we choose any leaf ω_μ of $T(\mu, k)$; by Lemma 5.2 we get

$$C(D) = \sum_{\mu \in L_{a-1}} \mathrm{wt}(P(\rho, \mu))\, \mathrm{wt}(T(\omega_\mu, n+1-a-k)) \sum_{\text{leaves } \omega \text{ of } T(\mu, k)} \frac{\mathrm{wt}(P(\mu, \omega))}{\mathrm{wt}(\mu)}$$

$$= \sum_{\mu \in L_{a-1}} \mathrm{wt}(P(\rho, \mu))\, \mathrm{wt}(T(\omega_\mu, n+1-a-k))\, \mathrm{wt}(T(\mu, k)).$$

At this stage we choose some element $\bar{\mu}$ of L_{a-1}, and Lemmas 5.1 and 5.3 yield

$$C(D) = \mathrm{wt}(T(\bar{\mu}, k)) \sum_{\mu \in L_{a-1}} \mathrm{wt}(P(\rho, \mu))\, \mathrm{wt}(T(\omega_\mu, n+1-a-k))$$

$$= C^*(V)\, C(D \backslash V). \quad \blacksquare$$

Two more definitions will be useful in the proofs of the lemmas. Recall that nodes are labelled by edges in $D^\#$. Let v be any node in T, say in level s. We define $J(v)$ to be the partial covering of $D^\#$ consisting of the edges that label the nodes in $P(\rho, v)$, and we define $W(v)$ as follows:

$$W(v) = \{ v : v \text{ is the terminal vertex of an edge in } J(v) \}.$$

Recall that, for all $i \in [n]$, c_i is the height of the ith column, and note that $\tau(s)$ is the initial vertex of the edge that labels v; thus, the possible destinations for vertex $\tau(s)$ in $D^\#$ are $[c_{\tau(s)}] \cup [n+1, n+x]$. The definition of a consistent ordering and the construction of the choice tree ensure that $W(v)$ will be a subset of these destinations and that $|W(v)| = s$.

Proof of Lemma 5.1. Let $\mathscr{P}$ be the set of all tree-paths from children of μ to leaves of $T(\mu, k)$, and let $\mathscr{P}'$ be the set of all tree-paths from children of μ' to leaves of $T(\mu', k)$. We want to show that

$$\sum_{P \in \mathscr{P}} \mathrm{wt}(P) = \sum_{P' \in \mathscr{P}'} \mathrm{wt}(P');$$

to do so we find a weight-preserving bijection from $\mathscr{P}$ to $\mathscr{P}'$.

Because $W(\mu)$ and $W(\mu')$ are subsets of $[n+x]$ that have the same cardinality, we can find a bijection

$$\phi : [n+x] \backslash W(\mu) \to [n+x] \backslash W(\mu');$$

moreover, we can take $\phi\,|\,[c_{\tau(a-1)}+1, n]$ to be the identity function, because neither $W(\mu)$ nor $W(\mu')$ contains a vertex of $[c_{\tau(a-1)}+1, n]$. Consequently, $\phi\,|\,V$ is also the identity function, because the independence of V ensures that $V \subset [c_{\tau(a-1)}+1, n]$.

We use ϕ to define the desired bijection $\Phi: \mathscr{P} \to \mathscr{P}'$. Suppose $P \in \mathscr{P}$. We identify P with its sequence of nodes; let the edges that label these nodes be

$$(\tau(a), t_1), (\tau(a+1), t_2), ..., (\tau(a+k-1), t_k).$$

We define $\Phi(P)$ to be the tree-path in $T(\mu', k)$ that is identified with the sequence

$$(\tau(a), \phi(t_1)), (\tau(a+1), \phi(t_2)), ..., (\tau(a-1+k), \phi(t_k)).$$

We should verify that this sequence corresponds to a path in $\mathscr{P}'$. By the construction of the tree, $t_1, t_2, ..., t_k$ are distinct elements of $[n+x]\backslash W(\mu)$, so the images under the bijection ϕ are certainly still distinct; therefore, the edges that label the nodes of $\Phi(P)$ are compatible with each other. We need to show that these edges are also compatible with the edges of $J(\mu')$ and that they are edges in $D^{\#}$. This follows from the construction of ϕ. If $t_i \in [c_{\tau(a-1)}+1, n]$, then $\phi(t_i) = t_i$, so $(\tau(a-1+i), \phi(t_i)) = (\tau(a-1+i), t_i)$. Since this edge is in the original sequence, it must be in $D^{\#}$, and it is compatible with the edges of $J(\mu')$ because $W(\mu')$ does not intersect $[c_{\tau(a-1)}+1, n]$. Otherwise, if $t_i \in [c_{\tau(a-1)}] \cup [n+1, n+x]\backslash W(\mu)$, then $\phi(t_i) \in [c_{\tau(a-1)}] \cup [n+1, n+x]\backslash W(\mu')$; in other words, $(\tau(a-1+i), \phi(t_i))$ is an edge of $D^{\#}$ compatible with the edges of $J(\mu')$.

To show that Φ is a bijection, wet let $\Psi := \mathscr{P}' \to \mathscr{P}$ be the map induced by ϕ^{-1}; it is straightforward to verify that $\Psi\Phi$ and $\Phi\Psi$ are identity maps.

Now we want to show that $\mathrm{wt}(\Phi(P)) = \mathrm{wt}(P)$. Let v be the jth node in P, and let v' be the jth node in $\Phi(P)$. Since the weight of a tree-path is the product of the weights of its nodes, we need only show that $\mathrm{wt}(v) = \mathrm{wt}(v')$. Suppose $\mathrm{wt}(v) = y$, so that the edge that labels v completes a cycle in $J(v)$. We claim that the set of vertices C of the cycle is a subset of V.

To show this, we rule out the successors indirectly, and then we rule out the predecessors. If a successor of V is in C, then it must be the initial vertex of an edge in the cycle. But no such edges can lie in $J(v)$, by the construction of T. Therefore, C is the disjoint union of $C \cap V_<$ and $C \cap V$. By the independence of V, there does not exist an edge from a vertex of $V_<$ to a vertex of V; it follows that either $C \cap V_<$ or $C \cap V$ must be empty. We know $C \cap V$ contains $\tau(a-1+j)$; therefore, $C \cap V_<$ is empty, and $C \cap V = C$, which means $C \subset V$, as claimed.

In particular, the initial vertices of the edges in the cycle are in V, so those edges label some subsequence of the sequence of nodes identified with P. Moreover, the terminal vertices of the edges that label this subsequence are also in V, so these same edges label the corresponding subsequence of nodes in $\Phi(P)$, because $\phi \mid V$ is the identity mapping. Therefore, the edge that labels v' completes the same cycle, and $\mathrm{wt}(v') = y$.

Last, we apply the above argument to Ψ to show that if $\mathrm{wt}(v') = y$, then $\mathrm{wt}(v) = y$; equivalently, if $\mathrm{wt}(v) = 1$, then $\mathrm{wt}(v') = 1$. ∎

Proof of Lemma 5.2. Let $\mathscr{P}$ be the set of all tree-paths from children of ω to leaves of T, and let $\mathscr{P}'$ be the set of all three-paths from children of ω' to leaves of T. As in the proof of Lemma 5.1, we construct a weight-preserving bijection $\Phi\colon \mathscr{P} \to \mathscr{P}'$, and we define Φ in terms of a bijection ϕ.

Both $J(\omega)$ and $J(\omega')$ are disjoint unions of maximal paths and cycles by the compatibility requirement in the construction of the T. Because the edges that label nodes in $P(\mu, \omega)$ do not generally correspond to the edges that label nodes in $P(\mu, \omega')$, $J(\omega)$ and $J(\omega')$ may not contain the same number of cycles. However, excluding the cycles but including isolated vertices, they will each contain $n + x - a + 1 - k$ maximal paths, by the argument we have in connection with Chung and Graham's definition of the cover polynomial. Moreover, by the construction of T, no destinations have been chosen for the successors of V in these partial coverings, so each vertex of $V_>$ will be the terminal vertex of a unique, possibly empty, maximal path. These are exactly the paths of $J(\omega)$ and $J(\omega')$ that might end up in a cycle, so we keep track of them in order to construct Φ.

Towards this end, we define two functions f, f': $V_> \to [n]$ by the following rule: $f(v)$ is the initial vertex of the maximal path in $J(\omega)$ that terminates at vertex v, and $f'(v)$ is the initial vertex of the maximal path in $J(\omega')$ that terminates at vertex v. We consider isolated vertices in the partial covering to be maximal paths; thus, if no edge of $J(\omega)$ terminates at $v \in V_>$, then $f(v) = v$.

Because $|W(\omega)| = a + k - 1 = |W(\omega')|$, we can construct a bijection

$$\phi\colon [n + x] \backslash W(\omega) \to [n + x] \backslash W(\omega').$$

We claim that we can require ϕ to satisfy two conditions: $\phi \mid [c_{\tau(a+k)} + 1, n]$ is the identity function, and $\phi(f(v)) = f'(v)$ for all $v \in V_>$. We can take ϕ to satisfy the first condition because $[c_{\tau(a+k)} + 1, n]$ does not intersect $W(\omega)$ nor $W(\omega')$. Similarly, $f(V_>)$ does not intersect $W(\omega)$, and $f'(V_>)$ does not intersect $W(\omega')$, because the elements of $f(V_>)$ and $f'(V_>)$ are initial vertices of maximal paths in $J(\omega)$ and $J(\omega')$. We also know that $|f(V_>)| = |f'(V_>)|$, because f and f' are injective; therefore, we can take ϕ to satisfy the second condition. We must also show that the two conditions do not conflict. Suppose $f(w) \in [c_{\tau(a+k)} + 1, n]$ for some $w \in V_>$. It follows from the independence of V that $V \subset [c_{\tau(a+k)}]$, so clearly $f(w) \notin V$. Because $J(\omega)$ is obtained from its subdigraph $J(\mu)$ by adding edges that originate at vertices of V, we can conclude $f(w)$ is the initial vertex of the maximal path in $J(\mu)$ that terminates at w. Consequently, $f(w)$ is also the initial vertex of the maximal path in $J(\omega')$ that terminates at w, because no edge of $J(\omega')$ can terminate at a vertex outside of $[c_{\tau(a+k-1)}]$. Therefore, $f'(w)=f(w)$, which means $\phi(f(w))=f(w)$, as desired.

Now we construct Φ. Let $P\in\mathscr{P}$. We identify P with the sequence of directed edges that label its nodes:

$$(\tau(a+k), t_1), (\tau(a+k+1), t_2), ..., (\tau(n), t_{n-a-k+1}).$$

We define $\Phi(P)$ to be the tree-path in $T(\omega', n+1-a-k)$ that is identified with the sequence

$$(\tau(a+k), \phi(t_1)), (\tau(a+k+1), \phi(t_2)), ..., (\tau(n), \phi(t_{n+1-a-k})).$$

As in the proof of Lemma 5.1, the construction of ϕ ensures that $\Phi(P)\in\mathscr{P}'$ and that Φ is a bijection.

Let v be the jth node of P for some $j\in[n+1-a-k]$, and let v' be the jth node of $\Phi(P)$. We want to show that $\mathrm{wt}(v)=\mathrm{wt}(v')$ and thus $\mathrm{wt}(P)=\mathrm{wt}(\Phi(P))$; as in the proof of Lemma 5.1, it will suffice to show that if $\mathrm{wt}(v)=y$, then $\mathrm{wt}(v')=y$.

So suppose $\mathrm{wt}(v)=y$; in other words, the edge that labels v completes a cycle in $J(v)$. We claim that the cycle can be decomposed into an alternating sequence of maximal paths in $J(\omega)$ and edges that are not contained in $J(\omega)$. These latter are the edges whose initial vertices are in $V_>$, the edges that label nodes in P. To verify this claim, we remove all such edges from the cycle. Because there is at least one, the label of v, what remains is no longer a cycle; rather, it is a set of disjoint paths in $J(\omega)$, possibly including isolated vertices of $V_>$. Each of these paths terminates at the initial vertex of one of the removed edges, in other words, a vertex of $V_>$. Moreover, these paths must be maximal in $J(\omega)$; otherwise, in $J(v)$, either an initial vertex of a path has indegree exceeding one, or the terminal vertex of the path has outdegree exceeding one, violating the compatibility condition. Consequently, each edge whose initial vertex is in $V_>$ terminates at a distinct vertex in $f(V_>)$.

Let $\{w_1, w_2, ..., w_s\}$ be the set of vertices of $V_>$ that lie in the cycle. We can assume without loss of generality that for $i\in[s-1]$ no vertex of $V_>$ lies on the directed path in the cycle from w_i to w_{i+1}, except the endpoints themselves. We can also assume that $w_s=\tau(a+k-1+j)$, the initial vertex of the edge that labels v. By the previous paragraph, the cycle decomposes into paths from $f(w_i)$ to w_i, for all $i\in[s]$, and edges in

$$\{(w_i, f(w_{i+1})): i\in[s-1]\}\cup\{(w_s, f(w_1))\}.$$

The first term in this union is the set of edges that label ancestors of v in P. In $J(\omega')$ there are paths from $f'(w_i)$ to w_i for all $i\in[s]$, and by the definitions of Φ and ϕ, $\{(w_i, f'(w_{i+1})): i \in [s-1]\}$ is a set of edges that label ancestors of v' in $\Phi(P)$. It follows that $(w_s, f'(w_1))$, the edge that labels v', completes a cycle in $J(v')$. ▮

Proof of Lemma 5.3. Recall that $D\backslash V$ is obtained from D by removing the vertices of V and any edge incident with a vertex of V. Strictly speaking, we must also relabel the vertices so that the vertex set is $[n-k]$. However, because this relabeling does not affect the cover polynomial, we will dispense with it; thus, the vertex set of $D\backslash V$ is $[n]\backslash V$. Let $D\backslash V^{\#}$ be the extension of $D\backslash V$ by the destination vertices $[n+1, n+x]$. Let T' be the choice tree associated with $D\backslash V^{\#}$, and let ρ' be the root of T'. Let $\mathscr{P}$ be the set of pairs of tree-paths (P_1, P_3), where P_1 is a tree-path from ρ to a node $\mu \in L_{a-1}$, and P_3 is a tree-path from a child of ω_μ to leaf of T. Let $\mathscr{P}'$ be the set of tree-paths from ρ' to a leaf of T'. We want to show that

$$\sum_{(P_1, P_3) \in \mathscr{P}} \mathrm{wt}(P_1)\, \mathrm{wt}(P_3) = \sum_{P' \in \mathscr{P}'} \mathrm{wt}\, P',$$

so we construct a bijection $\Phi: \mathscr{P} \to \mathscr{P}'$ for which $\mathrm{wt}(P_1)\, \mathrm{wt}(P_3) = \mathrm{wt}(\Phi(P_1, P_3))$.

Let (P_1, P_3) be a pair of tree-paths in T with the properties stated above. Let P_2 be the tree-path from the appropriate child of μ to ω_μ; thus, together, P_1, P_2, and P_3 encode a covering of $[n]$. In order to define Φ, we want to contract the edges that label P_2, because they are the edges that originate at vertices of V. Let J be the partial covering consisting of the edges that label P_2; as we have seen, J is a union of maximal paths and cycles. If v is the initial vertex of a maximal path in J, then let $g(v)$ be the terminal vertex of that path. Note that $g(v) \notin V$, because the path is maximal in J. Now define a function ϕ as follows:

$$\phi(v) = \begin{cases} g(v) & \text{if} \quad v \text{ is the initial vertex of a maximal path in } J \\ v & \text{otherwise.} \end{cases}$$

As in the previous lemmas, we identify P_1 and P_3 with sequences of edges,

$$(\tau(1), t_1), ..., (\tau(a-1), t_{a-1}) \qquad \text{and} \qquad (\tau(a+k), t_a), ..., (\tau(n), t_{n-k});$$

then we let $\Phi((P_1, P_2))$ be the path in T' identified with the sequence

$$(\tau(1), \phi(t_1)), ..., (\tau(a-1), \phi(t_{a-1})), (\tau(a+k), \phi(t_a)), ..., (\tau(n), \phi(t_{n-k})).$$

We verify that this sequence identifies a path in $\mathscr{P}'$. The initial vertices of the sequence are clearly distinct elements of $[n]\backslash V$. Now the compatibility in the construction of T tells us that t_i is not the terminal vertex of any edge in J for any $i \in [n-k]$. This has two consequences. First, if $t_i \in V$, then t_i is the initial vertex of a maximal path in J; therefore, $\phi(t_i) \notin V$. Since this clearly holds when $t_i \notin V$ as well, $\phi(t_i) \in [n+x] \backslash V$ for all $i \in [n-k]$. Second, if $t_i \in V$, then $g(t_i) \neq \phi(t_j)$ for any $j \neq i$; it follows that the $\phi(t_i)$ are distinct. The construction of ϕ also ensures that each of these edges lies in $D \backslash V^{\#}$.

To show that Φ is a bijection, we construct its inverse $\Psi: \mathscr{P}' \to \mathscr{P}$. Let $P' \in \mathscr{P}'$ be a tree-path from ρ' to a leaf λ'. Let μ' be the node in level L_{a-1} of this tree-path. We adjoin the vertices of V to $J(\mu')$ as isolated vertices. Because $D \backslash V^{\#}$ is a subdigraph of $D^{\#}$, $J(\mu')$ must equal $J(\mu)$ for some μ in L_{a-1} of T. Thus, we know $P_1 = P(\rho, \mu)$. We know P_2 as well, because for each μ, we have already chosen a leaf ω_μ of $T(\mu, k)$. As before, let J be the partial covering encoded by the nodes of P_2. We determine P_3 by inserting J into $J(\lambda')$. More precisely, suppose

$$(\tau(a+k), t_1), \ldots, (\tau(n), t_{n+1-a-k})$$

is the sequence of edges that label the descendants of μ' in P'. If t_i is the terminal vertex of a maximal path in J, let $g^{-1}(t_i)$ be the initial vertex of that path. Define a function ψ as follows:

$$\psi(v) = \begin{cases} g^{-1}(v) & \text{if } \quad v \text{ is the terminal vertex of a maximal path in } J \\ v & \text{otherwise.} \end{cases}$$

We define P_3 to be the path in $T(\omega_\mu, n+1-a-k)$ identified with the sequence of edges

$$(\tau(a+k), \psi(t_1)), \ldots, (\tau(n), \psi(t_{n+1-a-k})).$$

We leave it to the reader to verify that $(P_1, P_3) \in \mathscr{P}$ and that Φ and Ψ are inverses.

Finally, we observe that $\mathrm{wt}(P_1)\,\mathrm{wt}(P_2 = \mathrm{wt}(P')$; as before, it will suffice to show that the corresponding nodes in the paths have the same weight. Let v be a node in P_1 or P_3, and let v' be the node in P' corresponding to v. If $v \in P_1$ and the edge that labels v completes a cycle in $J(v)$, then the edge that labels v' completes the same cycle in $J(v')$, and conversely, because, as we observed before, $J(\mu)$ and $J(\mu')$ have the same edges. On the other hand, if $v \in P_3$ is labeled by an edge which completes a cycle, that cycle may contain maximal paths of J whose edges do not appear in $J(v')$. However, the construction of Φ essentially contracts those paths to their terminal vertices, so that the edge that labels v' still completes a cycle. The converse is similar: if the edge that labels v' completes a cycle in $J(v')$, then in $J(v)$ the construction of Ψ may expand some of the vertices of the cycle into maximal paths of J. However, the edges in these paths label ancestors of v, so the edge that labels v completes the cycle. ∎

6. APPLICATIONS OF THE PARTIAL FACTORIZATION THEOREM

The first corollary is the sufficient condition for $C^*(V)$ to be the cover polynomial of a skyline board that we mentioned in connection with Fig. 13. We define another subdigraph: let the *full restriction of D to V*, denoted $D \parallel V$, be the subdigraph induced by the vertices of V. In other words, we get $D \parallel V$ from $D|V$ by removing any vertices of $V_<$ and $V_>$ and any edges incident with those vertices.

COROLLARY 6.1. *Let D be the digraph of a skyline board, let τ be a consistent ordering of D, and let V be a subset of the vertices of D that is independent with respect to τ. For each vertex $u \in V_< \cup V_>$, suppose that either $(v, u) \in B$ for all $v \in V$ or that $(v, u) \notin B$ for all $v \in V$. Let l be the number of vertices of $V_< \cup V_>$ that satisfy the former condition. Then $C^*(V; x, y) = C(D \parallel V; x + l - |V_<|, y)$.*

Proof. Choose an arbitrary $\mu \in L_{a-1}$ from which to define $C^*(V) = \mathrm{wt}(T(\mu, k))$. Suppose $u \notin V$ and that $u \notin W(\mu)$. We saw in the proof of Lemma 5.1 that if for some $v \in V$ an edge (v, u) labels a node $\nu \neq \mu$ of $T(\mu, k)$, then $\mathrm{wt}(\nu) = 1$. If there exists any such edge (v, u), then by hypothesis there is an edge from every vertex of V to u, so we may regard u as a destination vertex in an extension of $D \parallel V$. The extra vertices in this extension are the x vertices adjoined to D originally, plus the l vertices with the property described above, minus the terminal vertices in $J(\mu)$ that originate at predecessors of V. Thus there are $x + l - |V_<|$ possibilities. ∎

The next corollary is a special case of a factorization theorem of Chung and Graham [2]:

COROLLARY 6.2. *Suppose $D = (V, E)$ is formed by joining the disjoint digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ with all the edges (v_1, v_2), where $v_1 \in V_1$ and $v_2 \in V_2$. Suppose also that D_1 and D_2 are digraphs of skyline boards. Then*

$$C(D) = C(D_1)\, C(D_2).$$

In Chung and Graham's version, D_1 and D_2 need not be digraphs of skyline boards.

Proof. The given conditions imply that V_2 is an independent subset with respect to some consistent ordering, with successors V_1 and no predecessors. Since $D \backslash V_2$ is D_1, Theorem 4.1 gives that $C(D) = C(D_1)\, C^*(D_2)$. Since V_2 satisfies trivially the condition of Corollary 6.1, we have $C^*(V_2) = C(D_2)$. ∎

The polynomial $C^*(V)$ also takes a simple form when V is a single vertex.

COROLLARY 6.3. *Let D be the digraph of a skyline board. If, for vertex j, $V=\{j\}$ is independent with respect to a consistent ordering τ, then*

$$C^*(V)=\begin{cases} x+c_j-\tau^{-1}(j)+1 & \text{if } c_j<j \\ x+c_j-\tau^{-1}(j)+y & \text{if } c_j\geqslant j. \end{cases}$$

Proof. By definition $C^*(V)=\operatorname{wt}(T(\mu,k))$ for some $\mu\in L_{a_1}$. Here $k=1$, so the weight of the tree is just the sum of the weights of the children of μ. There are $x+c_j$ vertices that are terminal vertices of edges in $D^{\#}$ originating at vertex j, $\tau^{-1}(j)-1$ of which are elements of $W(\mu)$, so there are $x+c_j-\tau^{-1}(j)+1$ such children. It follows from the proof of Lemma 5.1 that the only possible cycle is the edge from j to j, which is available if and only if $c_j\geqslant j$. ∎

We are now in a position to prove a result that we mentioned earlier: full columns and empty columns in the board correspond to linear factors of the cover polynomial.

COROLLARY 6.4. *Let D be the digraph of a skyline board. Let $A_n=\{i\colon c_i=n\}$ and $A_0=\{i\colon c_i\colon 0\}$. Then*

$$C(D)=(x+y)^{\overline{|A_n|}}\,x^{\underline{|A_0|}}C^*([n]\backslash(A_n\cup A_0)).$$

Proof. Let $V=[n]\backslash(A_0\cup A_n)$. This subset is independent with respect to any consistent ordering, so Theorem 4.1 gives $C(D)=C(D\backslash V)\,C^*(V)$. It remains to calculate the cover polynomial of $(D\backslash V)$, a board in which every column is either full or empty. Let n' be the number of columns in $D\backslash V$, and relabel its vertices with distinct numbers in $[n']$. If each column is empty, then $r_{0,0}=1$ and every other rook number is zero, so $C(D\backslash V)=x^{\underline{n'}}$, as given by the corollary in this case. Otherwise, let τ be a consistent ordering of $D\backslash V$, and let $j=\min\{i\colon c_{\tau(i)}=n'\}$. Note that here $|A_n|=n'-j+1$. We apply Theorem 4.1 again, with $\{\tau(j)\}$ as the independent set:

$$C(D\backslash V)=C([n']\backslash\{\tau(j)\})\,C^*(\{\tau(j)\}).$$

Induction gives us the first factor, and Corollary 6.3 gives us the second, yielding

$$\begin{aligned} C(D\backslash V)&=(x+y)^{\overline{|A_n|-1}}\,x^{\underline{|A_0|}}\cdot(x+n'-j+y)\\ &=(x+y)^{\overline{|A_n|}}\,x^{\underline{|A_0|}}. \end{aligned}$$ ∎

One of the referees of this paper pointed out an elegant proof of the calculation of $C([n]\backslash V)$ in this corollary that avoids the machinery of Theorem 4.1. Instead it is based on Chung and Graham's factorization theorem and a reciprocity theorem that was independently discovered by Gessel [5] and Chow [1]. Chow actually gets it as a corollary of a more general result involving his "path-cycle" symmetric function, and Haglund [7] generalizes Chow's result even further.

Given a diagraph board B, we define the *complement* of B, denoted $\bar{B}$, to be $[n]\times[n]\backslash B$. Let $\bar{D}$ denote the digraph corresponding to $\bar{B}$. We state the reciprocity theorem without proof.

THEOREM 6.5. $C(\bar{D}; x, y) = (-1)^n\, C(D; -x-y, y)$.

Alternate Proof that $C(D\backslash V) = (x+y)^{\overline{|A_n|}} x^{\underline{|A_0|}}$ *in Corollary* 6.4. Let D_1 and D_2 be the subdigraphs of $D\backslash V$ induced by A_n and A_0: D_2 is the empty board E_{A_0} and D_1 is a "full" board, which we can represent as $\bar{E}_{A_n}$. These digraphs clearly satisfy the conditions of Corollary 6.2, so we get $C(D\backslash V) = C(E_{A_0})\, C(\bar{E}_{A_n})$. We already noted in the first proof of Corollary 6.4 that $C(E_m; x, y) = x^{\underline{m}}$ for any positive integer m. Theorem 6.5 then gives that $C(\bar{E}_m; x, y) = (-1)^m\, C(E_m; -x-y, y) = (-1)^m (-x-y)^{\underline{m}} = (x+y)^{\overline{m}}$. The result follows. ∎

We also get alternate proofs of Theorems 3.2 and 3.3 as corollaries:

Alternate Proof of Theorem 3.2. If $c_i < i$ for all $i \in [n]$, then there are no cycles, and the result reduces to Theorem 3.1. Otherwise, there exists a column number j such that $c_j \geq j$, and we can take j to be minimal. Let τ be the identity permutation; clearly this is a consistent ordering for B. Let $V_< = [j-1]$, $V = \{j\}$, and $V_> = [n]\backslash[j]$. The increasing column heights and the choice of j ensure that V is independent with respect to τ. By Corollary 6.3, $C^*(V) = x + c_j - \tau^{-1}(j) + y = x + c_j - j + y$. Since $D\backslash V$ corresponds to an increasing Ferrers board with $n-1$ columns, the result follows by induction. ∎

Alternate Proof of Theorem 3.3. For a decreasing Ferrers board, the above proof goes through almost word for word. We need only the following modifications: we take j to be maximal, we let $\tau(i) = n+1-i$, and we switch $V_<$ and $V_>$ in the partition. ∎

7. PARTIAL FACTORIZATION OF COLUMN-PERMUTED STAIRCASE BOARDS

A certain kind of increasing Ferrers boards has been of interest in the literature on rook polynomials, because its rook numbers are Stirling numbers of the second kind [9, p. 75]: if B is the board with column heights $c_i = i-1$ for all i in $[n]$, then $r_i = \left\{ {n \atop n-i} \right\}$, the number of set partitions of $[n]$ into $n-i$ blocks. Given a compatible i-subset of B, we define a set partition by letting j and k be in the same block of the partition if (j, k) or (k, j) is one of the cells in the subset. It is straightforward to verify that this gives a bijection. In Fig. 14 we give an example with $n=7$.

We next investigate the cover polynomial of the skyline boards that arise from permuting the columns of these boards. We define *the staircase board*, denoted R_n, to be the Ferrers board with column heights $c_i = i$ for every $i \in [n]$. Notice that we have modified the above to including the diagonal. If we add an empty column to the left and an empty row on the top, we increase n by 1 without changing the rook numbers, implying $r_i(R_n) = \left\{ {n+1 \atop n+1-i} \right\}$. Although this formula is slightly less elegant than before, for our purpose it will be more natural to include the diagonal.

Given a permutation σ, observe that the column heights of $\sigma(R_n)$ are $c_i = \sigma(i)$, for each $i \in [n]$, so the board consists of the cells that lie on or below the graph of σ. Similarly, in the digraph interpretation of the board, we can simplify the picture by omitting most of the directed edges as follows: let $D(\sigma)$ be the digraph on n vertices with a directed edge from i to $\sigma(i)$, for each $i \in [n]$. Thus $D(\sigma)$ is the usual functional digraph of σ; we shall call it the *skeleton of* $\sigma(R_n)$.

In the Fig. 15, $n=3$ and $\sigma = (123)$, so that $c_1 = \sigma(1) = 2$, $c_2 = \sigma(2) = 3$, and $c_3 = \sigma(3) = 1$.

We shall define an equivalence relation on the vertices of $\sigma(R_n)$ such that each equivalence class corresponds to a factor of the cover polynomial. In fact, it will be possible to further reduce each such factor, provided it is not already linear. It turns out that we can illustrate the equivalence classes with a simple drawing of the skeleton: we line up the vertices horizontally in order from left to right, draw edges which are directed from left to right

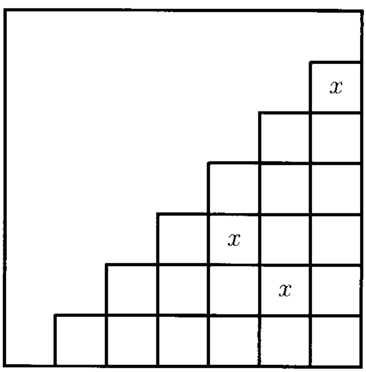


FIG. 14. The 3-subset corresponds to $\{2, 6, 7\} \cup \{3, 5\} \cup \{1\} \cup \{4\}$.

in an arc above the line, and draw edges which are directed from right to left in an arc below the line. The non-intersecting pieces in such a drawing will correspond to the equivalence class. In Fig. 15, the digraph $D(\sigma)$ is drawn in this way: it has one equivalence class. Consider the example in Fig. 16 with two equivalence classes.

The cycle (23) is an entire equivalence class, which we can regard as the permutation σ_2 on two letters; similarly, the cycle (145) becomes the permutation σ_1 on three letters. Notice that we already considered $\sigma_1(R_3)$ in Fig. 15, and we calculated its cover polynomial in Fig. 10.

Let us now define the equivalence relation $\equiv$ for a given σ. First we define a relation $\sim$ on cycles of σ: $C_1 \sim C_2$ if there exists $i_1, j_1 \in C_1$ and $i_2, j_2 \in C_2$ such that $i_1 < i_2 < j_1 < j_2$ or $i_2 < i_1 < j_2 < j_1$. Let $\equiv$ be the reflexive transitive closure of this relation. We can regard $\equiv$ as an equivalence relation on the vertices, as well as the cycles containing them.

If we draw the skeleton in the manner described above, then, it is indeed true that $C_1 \sim C_2$ if and only if their drawings intersect. Since we will not use the drawing of the skeleton of σ in the proof of the factorization theorem, except as a means of description and motivation, we omit the formal proof of this fact. Instead we will formalize as lemmas two geometric intuitions suggested by the drawing. First, we state the factorization theorem.

THEOREM 7.1. *Let σ be a permutation on n letters, and let $V_1, V_2, ..., V_m$ be the equivalence classes of the vertices under $\equiv$, with cardinalities $k_1, k_2, ..., k_m$, respectively. For each $i \in [m]$, let σ_i be the restriction of σ to V_i, with the vertices relabeled in the same order as members of $[k_i]$. Then*

$$C(\sigma(R_n)) = \prod_{i=1}^{m} C(\sigma_i(R_{k_i})).$$

Note that the σ_i are well-defined permutations, because each equivalence class is a union of cycles.

The geometric intuition for the first lemma is that the interval "spanned" by sequence of related cycles is "covered" by the cycles:

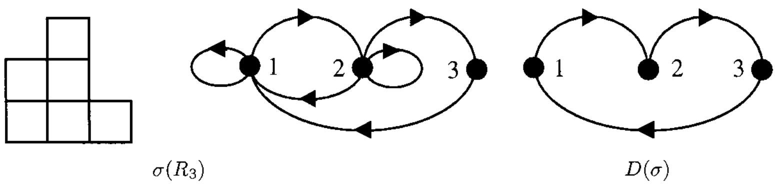


FIGURE 15

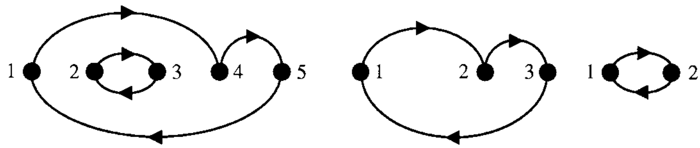


FIG. 16. (a) $\sigma=(145)(23)$; (b) $\sigma_1=(123)$; (c) $\sigma_2=(12)$.

$$\begin{aligned} C(\sigma(R_5)) &= [(x+1)(x^2+xy+x+1)][(x+y)(x+1)] \\ &= C(\sigma_1(R_3))\, C(\sigma_2(R_2)). \end{aligned}$$

LEMMA 7.2. *For a given permutation, let $C_0 \sim C_1 \sim \cdots \sim C_s$ be a sequence of cycles, and let $V=\bigcup_{i=0}^{s} C_i$. Then*

$$\bigcup_{i=0}^{s} [\min C_i, \max C_i] = [\min V, \max V].$$

Proof. We proceed by induction on s. The case $s=0$ is immediate; suppose $s>0$. If $\min V \in C_j$ and $\max V \in C_k$ with $|j-k|<s$, then by induction

$$[\min V, \max V] = \bigcup_{i=\min\{j,k\}}^{\max\{j,k\}} [\min C_i, \max C_i] \subset \bigcup_{i=0}^{s} [\min C_i, \max C_i].$$

The reverse inclusion is clear. So suppose without loss of generality that $\min V \in C_0$ and $\max V \in C_s$. Let $V' = \bigcup_{i=1}^{s} C_i$. It follows from the definitions of V' and $\sim$ that $\min V' \leqslant \min C_1 < \max C_0$. So we can conclude by induction that

$$\begin{aligned} \bigcup_{i=0}^{s} [\min C_i, \max C_i] &= [\min C_0, \max C_0] \cup [\min V', \max V'] \\ &= [\min C_0, \max V']. \end{aligned}$$

It only remains to observe that $\min C_0 = \min V$ and $\max V' = \max V$. ∎

The second lemma builds upon the first by considering the relative positions of the equivalence classes. The geometric intuition is that for any two equivalence classes, either one is completely to the right of the other, or one is nested completely inside the other. Hence, there exists at least one "innermost" component, which will consists of consecutive vertices:

LEMMA 7.3. *For a given permutation, there exists an equivalence class of consecutive vertices.*

Proof of Lemma 7.3. Define a relation $\preccurlyeq$ on equivalence classes as follows: $V_i \preccurlyeq V_j$ if $\min V_j \leqslant \min V_i \leqslant \max V_i \leqslant \max V_j$. Clearly $\preccurlyeq$ is a partial order. There must exist a class V which is minimal in this ordering. We will show that V consists of consecutive vertices.

We argue by contradiction. Suppose there exists a vertex z such that $z \notin V$ and $\min V < z < \max V$. By Lemma 7.2, $\min C < z < \max C$ for some C in V. Let Z be the equivalence class containing z. The minimality of V in $\preccurlyeq$ implies that either $\min Z < \min V$ or $\max V < \max Z$; in either case there exists a vertex $\bar{z}$ such that $\bar{z} \notin [\min C, \max C]$ and $z \equiv \bar{z}$. Therefore, there exists a sequence $C_0 \sim C_1 \sim \cdots \sim C_t$ of cycles of Z such that $C_0 \cap [\min C, \max C] \neq \emptyset$ and $C_t \cap ([n] \backslash [\min C, \max C]) \neq \emptyset$. Furthermore, among all such sequences, we can take one with t minimal. Let $\tilde{z}$ be an element of $C_0 \cap [\min C, \max C]$. Now $V \neq Z$ implies that $C \nsim C_0$; therefore, it cannot be the case that $\min C_0 < \min C < \tilde{z} < \max C$, nor $\min C < \tilde{z} < \max C < \max C_0$. The only other configuration is $\min C < \min C_0 < \max C_0 < \max C$. But this contradicts the minimality of t: because $C_0 \sim C_1$, there exists a vertex $y \in C_1$ such that $\min C_0 < y < \max C_0$, implying that $y \in C_1 \cap [\min C, \max C]$. ∎

Proof of Theorem 7.1. We proceed by induction on m, the number of equivalence classes. By Lemma 7.3, there exists an equivalence class V of consecutive vertices. Let $V = [a, a+k-1]$ for some $a, k \in [n]$, and suppose $V = V_1$ without loss of generality. Let σ_V and $\sigma_{\bar{V}}$ be the restrictions of σ to V and $[n] \backslash V$, will appropriate relabeling of the vertices. By induction on m,

$$C(\sigma_{\bar{V}}(R_{n-k})) = \prod_{i=2}^{m} C(\sigma_i(R_{k_i})).$$

Therefore, it will suffice to show that

$$C(\sigma(R_n)) = C(\sigma_V(R_k))\, C(\sigma_{\bar{V}}(R_{n-k})).$$

The unique consistent ordering of $\sigma(R_n)$ is $\tau(i) = \sigma^{-1}(i)$. We verify the three conditions for the independence of V with respect to τ. We have that V is consecutive with respect to τ, because, as for any equivalence class, $\sigma^{-1}(V) = V$; thus $\tau(V)$ is a set of consecutive vertices. Furthermore, $V_< = \tau([1, a-1])$ and $V_> = \tau([a+k, n])$, so it follows from the definition of $\sigma(R_n)$ that the sets of column heights of vertices in $V_<$, V, and $V_>$ are $[1, a-1]$, $[k, a+k-1]$, and $[a+k, n]$. Therefore, if u is a predecessor of V, then $\sigma(u) < a$, so that $\sigma(u) \notin V$. Similarly, for any successor w, $\sigma(w) \in V$.

Therefore, we may apply Theorem 4.1 to get $C(\sigma(R_n)) = C^*(V)$ $C(\sigma_{\bar{V}}(R_{n-k}))$. The properties described above also justify an application of Corollary 6.1. In particular, for all $v \in V$ and $u \in V_<$, we know (v, u) is an edge in $\sigma(R_n)$ because $u < a \leqslant \sigma(v)$; similarly, for all $w \in V_>$, (v, w) is not an edge because $\sigma(v) < a + k \leqslant w$. Hence, Corollary 6.1 gives $C^*(V) = C(R_n \parallel V; x + l - |V_<|, y) = C(R_n \parallel V)$, because $l = |V_<|$ here. It only remains to observe that $R_n \parallel V = (\sigma_V(R_k))$, because both are equivalent to the board obtained by restricting R_n to the columns of V and then removing the bottom $a - 1$ rows. ∎

Theorem 7.1 reduces the question of the factorability of the cover polynomial of column-permuted staircase boards to the case where the vertices are equivalent under $\equiv$. In Fig. 16, we see that $x + y$ is a factor of both $C(\sigma_1(R_3))$ and $C(\sigma_2(R_2))$. This is true in general, by Corollary 6.4, because every column-permuted staircase board has one full column. We also see that $C(\sigma_1(R_3))$ has an irreducible factor that is quadratic in x, so in general the cover polynomial of a column-permuted staircase board will not factor completely. On the other hand, $C(\sigma_2(R_2))$ does factor completely, and there are other column-permuted staircase boards with this property. For example, let $n = 5$ and $\sigma = (1\ 5\ 4\ 3\ 2)$, as depicted in Fig. 17.

This example readily generalizes.

THEOREM 7.4. *If* $\sigma = (1\ n\ n-1 \cdots 3\ 2)$ *then* $C(\sigma(R_n)) = (x + y)(x + 1)^{n-1}$.

Proof. There are no empty columns in the board, and Column 1 is the only full column. Let $V = [2, n]$; then Corollary 6.4 gives $C(\sigma(R_n)) = (x + y)\ C^*(V)$. Moreover, since $V_< \cup V_> = \{1\}$ and there is an edge from every vertex of V to vertex 1, Corollary 6.1 applies, with $m = 1$ and $|V_<| = 0$, giving $C^*(v) = C(\sigma(R_n) \parallel V; x + 1, y)$. But $\sigma(R_n) \parallel V$ is simply the increasing Ferrers board with column heights $c_i = i - 1$ for $i \in [n-1]$. By Theorem 3.2, $C(\sigma(R_n) \parallel V) = x^{n-1}$, which means that $C^*(V) = (x + 1)^{n-1}$. ∎

Theorems 7.15 and 7.18 immediately give another result.

COROLLARY 7.5. *Let* σ *be a permutation with* c *cycles such that the restriction of* σ *to every equivalence class under* $\equiv$ *is a cycle of the form*

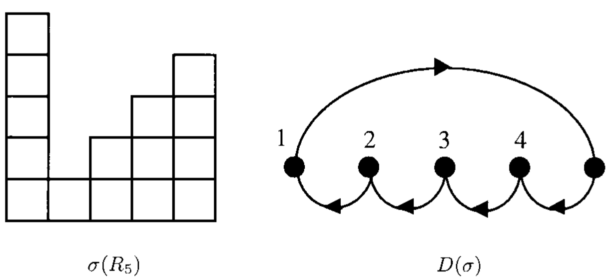


FIG. 17. $C(\sigma(R_5)) = (x + y)(x + 1)^4$.

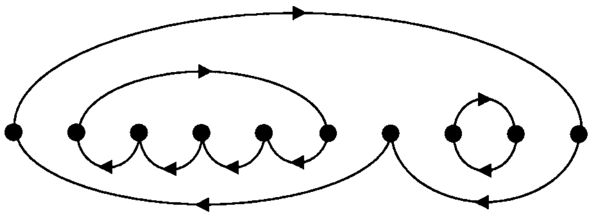


FIG. 18. $C(\sigma(R_{10})) = (x+y)^3(x+1)^7$.

$(a_1 a_k a_{k-1} \cdots a_3 a_2)$, *where* $a_1 < a_2 < \cdots < a_{k-1} < a_k$. *Then* $C(\sigma(R_n)) = (x+y)^c\,(x+1)^{n-c}$.

The permutations described in Corollary 7.5 correspond bijectively with "non-crossing partitions" of $[n]$, which are defined in essentially the same way as the classes of $\equiv$, ignoring the cycle structure. For a given n, the number of these is known to be $(1/(n+1))\binom{2n}{n}$, the nth Catalan number [8, Corollary 4.2]. Moreover, the number of permutations of this form with k cycles equals the number of non-crossing partitions of $[n]$ with k blocks, which is $(1/n)\binom{n}{k}\binom{n}{k-1}$, a Narayana number [8, Corollary 4.1].

To illustrate Corollary 7.5, it suffices to draw a skeleton in which the pieces are permutations of the form given in Theorem 7.4. In the example shown in Fig. 18, $n = 10$ and $\sigma = (1\ 10\ 7)(2\ 6\ 5\ 4\ 3)(8\ 9)$.

8. CONJECTURES

In the appendix we list all permutations which induce a single equivalence class under $\equiv$ for $n \leqslant 5$, according to the cover polynomials of the column permutation of the staircase board. If a permutation induces a single equivalence class, we conjecture that the cover polynomial factors completely only if the permutation takes the form $(1\ n\ n-1 \cdots 3\ 2)$. These are the only column permutations of the staircase board in which only one column reaches the diagonal. Equivalently, the permutation has one *excedance*, that is, a number $i \in [n]$ such that its image under the permutation exceeds i. We conjecture that if two permutations induce a single equivalence class, and if they induce the same cover polynomial, then they have the same cycle structure, and corresponding cycles contain the same number of excedances. Moreover, the number of factors of the cover polynomial roughly increases as the number of excedances decreases; it would be interesting to find a more precise connection. At the moment, the extent of our insight is that appropriate repeated applications of Theorem 4.1 always seem to produce the irreducible factorization. We conjecture that this is true not only for column-permuted staircase boards, but indeed for any skyline board.

APPENDIX:COLUMN-PERMUTED STAIRCASE BOARDS

These are the permutations with $n \leqslant 5$ in which all of the vertices are equivalent under $\equiv$ grouped according to the cover polynomial of the column-permuted staircase board.

Cover polynomial	*Permutations*
$(x+y)$	(1)
$(x+y)(x+1)$	(1 2)
$(x+y)(x+1)^2$	(1 3 2)
$(x+y)(x^2+(y+1)\,x+1)$	(1 2 3)
$(x+y)(x+1)^3$	(1 4 3 2)
$(x+y)(x+1)(x^2+(1+y)\,x+1)$	(1 4 2 3), (1 3 2 4), (1 2 4 3), (1 3 4 2)
$(x+y)(x^3+(1+2y)\,x^2+(1+y+y^2)\,x+1)$	(1 2 3 4)
$(x+y)(x^3+(2+y)\,x^2+(2+y)\,x+y)$	(1 3) (2 4)
$(x+y)(x+1)^4$	(1 5 4 3 2)
$(x+y)(x+1)^2\,(x^2+(1+y)\,x+1)$	(1 2 5 3 4), (1 4 5 3 2), (1 5 4 2 3), (1 5 3 2 4), (1 3 5 4 2), (1 5 3 4 2), (1 3 2 5 4), (1 5 2 4 3), (1 4 5 3 2), (1 4 3 2 5)
$(x+y)(x^2+(1+y)\,x+1)$	(1 2 4 5 3), (1 4 5 2 3), (1 4 2 3 5), (1 3 4 2 5), (1 2 5 3 4)
$(x+y)(x+1)(x^3+(1+2y)\,x^2 + (1+y+y^2)\,x+1)$	(1 2 4 3 5), (1 2 3 5 4), (1 3 2 4 5), (1 3 4 5 2), (1 5 2 3 4)
$(x+y)(x^4+(2+2y)\,x^3+(4+y+y^2)\,x^2 + (3+2y-y^2)\,x+1)$	(1 3 5 2 4)
$(x+y)(x^4+(1+3y)\,x^3+(1+2y+3y^2)\,x^2 + (1+y+y^2+y^3)\,x+1)$	(1 2 3 4 5)
$(x+y)(x^4+(3+y)\,x^3+(4+2y)\,x^2 + (2+2y)\,x+1)$	(1 4 2 5 3)
$(x+y)(x+1)(x^3+(2+y)\,x^2 + (2+y)\,x+y)$	(1 4 3)(2 5), (1 4)(2 5 3), (1 4 2)(3 5) (1 5 3)(2 4), (1 3)(2 5 4)
$(x+y)(x^4+(2+2y)\,x^3+(3+2y+y^2)\,x^2 + (2+2y)\,x+y)$	(1 3 4)(2 5), (1 4)(2 3 5), (1 2 4)(3 5) (1 3 5)(2 4), (1 3)(2 4 5)

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